$$dq_l/dt = u_l(\mathbf{q}) + (1/2)q_s(\partial u_s/\partial y_l)(\mathbf{q}) + (a^2/6\mu)\,(\partial p^0/\partial y_l)(\mathbf{q}), \tag{15}$$

where t is the time.

Equations (15) which have been derived describe the motion of the center of a particle in a nonuniform flow of a viscous incompressible liquid.

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THEORY OF TURBULENT MIXING AT THE INTERFACE OF FLUIDS IN A GRAVITY FIELD

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UDC 532.517.4

The theory of turbulent mixing at the interface of two media in accelerated motion was constructed in [1], and an approximate solution was given for incompressible fluids. The time variation of kinetic energy was neglected in the equation of balance for the kinetic energy of the turbulent motion. In [2] the characteristic turbulent velocity is averaged over the mixing region. This allows the initial equations to be solved allowing for the time variation of kinetic energy. It turns out that the resulting density profile roughly coincides with the profile of [1] within a wide range of variation of the initial density differential. In the present paper the equations for the mixing of incompressible fluids are studied in their complete form. It is established that the solutions of [1, 2] are applicable within a limited region, valid for small density ratios. The resulting solution is analyzed qualitatively, and it is shown that the density gradient at the mixing front is discontinuous. The dependence of the solution on two empirical constants is investigated. An approximate choice of the values of these constants is made on the basis of the theoretical considerations of [2, 3], and by comparison with the solution of [1]. The mixing asymmetry is found numerically as a function of the initial density differential. Quantitative characteristics of the solution are illustrated in graphs.

1. Formulation of the Problem

In order to describe the turbulent mixing of two substances of constant densities ρ_1 and ρ_2 situated in a gravity field g_0 a semiempirical theory is constructed. A characteristic turbulence velocity v and characteristic turbulence length l are introduced. An energy balance equation for the turbulence velocity v is constructed from dimensional considerations [1]:

$$\partial \rho v^2/2\partial t + \nu\rho v^3/l = \rho l v \omega^2. \tag{1.1}$$

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The first term on the left-hand side denotes the time variation of kinetic energy, while the second term with the unknown factor ν describes the energy dissipation of the turbulent motion. The right-hand side of the equation, which defines the whole process, is written from dimensional considerations [1]. Here ω is the growth frequency of short-wave perturbations obtained on investigating the stability of arbitrary one-dimensional flow,

$$\omega^2 = g_0 \partial \ln \rho / \partial x > 0.$$

This last expression is assumed to be zero if $g_0 \partial \ln \rho / \partial x < 0$. This is the stable case when turbulent mixing does not arise. To close the system of equations it is assumed that

$$l = \alpha L, \tag{1.2}$$

where L is the width of the mixing region, α is a constant, which, like ν, is to be determined experimentally.

If incompressibility is assumed, the equation for the density of the mixture can be written in the form

$$\partial \rho / \partial t = (\partial / \partial x) D \partial \rho / \partial x, \quad D = lv. \tag{1.3}$$

If necessary the concentration of the lighter substance $c = \rho_2 / \rho$ can be introduced.

The term with the time derivative in Eq. (1.1) was rejected in [1], while in [2] the turbulent velocity v was assumed to be independent of the spatial variable, and Eq. (1.1) was averaged over the mixing region. All this simplified the solution of the problem, although it led to appreciable discrepancies in a series of cases.

2. Self-Similar Nature of the Motion

The problem of the mixing of two incompressible fluids with constant densities ρ_1 and ρ_2 is self-similar. It is convenient to introduce the new time variable τ:

$$d\tau = l dt. \tag{2.1}$$

Equations (1.1), (1.4) assume the form

$$\begin{gathered} \partial v^2 / 2\partial \tau + (v^2/2) \partial \ln \rho / \partial \tau + \nu v^3 / l^2 = v g_0 \partial \ln \rho / \partial x; \\ \partial \rho / \partial \tau = (\partial / \partial x) v \partial \rho / \partial x. \end{gathered} \tag{2.2}$$

We now introduce the dimensionless variables

$$\begin{gathered} \lambda = x g_0^{-1/3} \tau^{-2/3}; \quad \tau = (1/27) \alpha^3 (\lambda_1 - \lambda_2)^3 g_0 t^3; \\ v = \tau^{1/3} g_0^{2/3} \zeta(\lambda); \quad \rho = \rho_1 \delta(\lambda). \end{gathered} \tag{2.3}$$

Here λ_1 and λ_2 correspond to the right- and left-hand sides of the mixing front. The second relation of Eqs. (2.3), expressing the new variable τ in terms of the time t, was obtained by integrating Eq. (2.1), making use of Eq. (1.2) and the expression for the width L:

$$L = g_0^{1/3} \tau^{2/3} (\lambda_1 - \lambda_2).$$

The acceleration g_0 is constant. We could assume that

$$g = g_0 t^{\beta},$$

in which case the powers in Eqs. (2.3) would be different. However, in this treatment we cannot assume that the acceleration is an arbitrary function of time as in [1, 2], since the problem under consideration would not then be self-similar.

Substitution of Eqs. (2.3) into Eqs. (2.2) leads to the system of ordinary differential equations

$$\begin{gathered} (2/3)\lambda \zeta' + (1 + \lambda \zeta / 3) \delta' / \delta - (1/3) \zeta - \zeta^2 / A = 0; \\ -(2/3) \lambda \delta' = (\zeta \delta')', \quad A = (\alpha^2 / \nu)(\lambda_1 - \lambda_2)^2 \end{gathered} \tag{2.4}$$

with the boundary conditions

$$\lambda = \lambda_1:\ \zeta_1 = 0;\ \delta_1 = 1; \qquad (2.5)$$

$$\lambda = \lambda_2:\ \zeta_2 = 0;\ \delta_2 = 1/n = \rho_2/\rho_1. \qquad (2.6)$$

These last relations result from the obvious conditions at the mixing front, initial densities ρ_1 and ρ_2, and a zero turbulent velocity $v_1 = v_2 = 0$.

A feature of the resulting system (2.4) is the fact that its coefficient A is a function of values of the unknown constants α and ν, which must be determined experimentally.

The order of the system (2.4) can be lowered if we introduce the new variable y:

$$y^2 = \delta'/\delta. \qquad (2.7)$$

It is obvious that the system (2.4) reduces to a system of two first-order equations:

$$\begin{gathered}(1/3)\zeta + (1/A)\zeta^2 - (2/3)\lambda\zeta' = (1 - \lambda\zeta/3)y^2;\\ -[(2/3)\lambda - \zeta'] = \zeta(y^2 - 2y'/y).\end{gathered} \qquad (2.8)$$

It is shown in Appendix 1 that the required solution must pass through the points I and II,

$$\text{I}\ \ (\lambda = \lambda_1;\ y_1 = (2/3)\lambda_1;\ \zeta_1 = 0); \qquad (2.9)$$

$$\text{II}\ \ (\lambda = \lambda_2;\ y_2 = -(2/3)\lambda_2;\ \zeta_2 = 0). \qquad (2.10)$$

Both points I and II are singular points of the system (2.8) and are saddle-point singularities. The required solution has the following expansions at point I:

$$y = \frac{2}{3}\lambda_1 + \left(\frac{1}{4} - \frac{\lambda_1^3}{27}\right)(\lambda - \lambda_1) + \ldots$$

$$\zeta = -\frac{2}{3}\lambda_1(\lambda - \lambda_1) + \left(-\frac{1}{12} + \frac{\lambda_1^3}{9}\right)(\lambda - \lambda_1)^3 + \ldots,$$

at point II:

$$y = -\frac{2}{3}\lambda_2 + \left(-\frac{1}{4} + \frac{\lambda_2^3}{27}\right)(\lambda - \lambda_2) + \ldots$$

$$\zeta = -\frac{2}{3}\lambda_2(\lambda - \lambda_2) + \left(-\frac{1}{12} + \frac{\lambda_2^3}{9}\right)(\lambda - \lambda_2)^2 + \ldots.$$

The values of λ_1 and λ_2 are unknown. An additional integral relation is obtained on integrating Eq. (2.7) taking the boundary conditions (2.5), (2.6) into account:

$$\int_{\lambda_2}^{\lambda_1} y^2 d\lambda = \ln n\,. \qquad (2.11)$$

This relation results in a closed problem and enables us to find a unique solution. This solution is constructed by numerical integration from points I and II in such a way as to ensure that one of the required functions ζ, for example, joins up and is continuous at the singular point $\lambda = 0$. For a fixed value of λ_1 this can be achieved by the choice of λ_2. Because of the singular nature of the point at $\lambda = 0$ the value of the other function y obtained at this point is continuous. The values of ζ and y are connected by the following relation which results from Eq. (2.8) when $\lambda = 0$:

$$y_0^2 = (1/3)\zeta_0 + (1/A)\zeta_0^2.$$

The initial point of integration λ_1 is found by iteration until the integral relation (2.11) is satisfied.

3. Choice of Empirical Constants

It has already been noted that the self-similar solution depends on the parameter A which contains the ratio of the empirical constants α^2/ν in its definition. The value of α^2/ν can be chosen by comparing the dimensionless density profiles obtained experimentally and theoretically. This necessitates a change to the new variables $\bar{\lambda}$, $\bar{\delta}$:

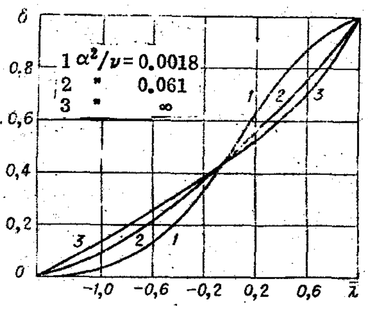

Fig. 1

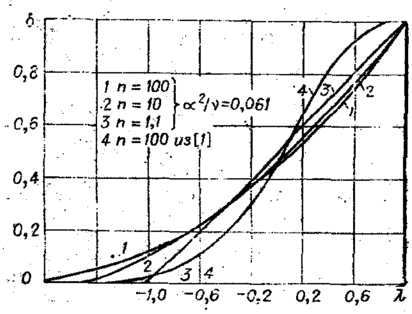

Fig. 2

$$\bar{\delta} = (n\delta - 1)/(n - 1) = (\rho - \rho_2)/(\rho_1 - \rho_2);$$

$$\bar{\lambda} = \lambda/\lambda_1 = r/r_1(t),$$

where $r_1(t)$ corresponds to the mixing front on the side of the heavy substance. The theoretical profile varies within wide limits depending on the value of α^2/ν. A comparison of density profiles $\bar{\delta}(\bar{\lambda})$ for n = 10 is given in Fig. 1.

The value of the constant α^2/ν can also be obtained theoretically using the results of [2, 3]. It was shown in [2] that on eliminating acceleration the width of the mixing region develops according to the power law

$$L \sim t^n, \quad n = 1/(1 + \nu/8\alpha^2\eta_0^2). \tag{3.1}$$

The quantity η_0 determines the effective width. If the effective width is defined as the zone where $0.1 \leq \bar{\delta} \leq 0.9$, then $\eta_0 = 0.906$.

It has been shown in the theoretical treatment [3] that the turbulence decreases according to a 2/7 law, i.e., the characteristic turbulence length l depends on time as follows:

$$l \sim t^{2/7}. \tag{3.2}$$

Equating the powers in Eqs. (3.1), (3.2), we obtain an expression for α^2/ν:

$$\alpha^2/\nu = 1/20\eta_0^2 = 0.061.$$

It follows from Fig. 1 that starting from some value of the parameter α^2/ν the dimensionless profile $\bar{\delta}$ is only weakly dependent on the accuracy with which the parameter is determined. Within the wide range $0.061 < \alpha^2/\nu < \infty$ the difference between curves 2 and 3 in Fig. 1is not appreciable. Some features of the case $\alpha^2/\nu = \infty$ are set out in Appendix 2.

The other empirical constant α is chosen by comparing the theoretical width L with its experimental value. We propose to do this in practice by measuring the width L_1 taken from the initial position of the interface in the direction of the heavy substance. It is preferable to compare L_1 and not L_2, since the theoretical density profile has a sharp front in the direction of the heavy substance, while in the direction of the light substance the front could be smeared out, especially for large values of n (Fig. 2).

The values of λ_1 and λ_2 must be known in order to determine the constant α in the formula for L_1:

$$L_1 = (\alpha^2/9)(\lambda_1 - \lambda_2)^2\lambda_1 g_0 t^2$$

Graphs of the quantities $-\lambda_1/\lambda_2$ and λ_1^3 are given in Fig. 3 as functions of ln n for $\alpha^2/\nu = 0.061$.

We now evaluate α, comparing the total width L

$$L = (\alpha^2/9)(\lambda_1 - \lambda_2)^3 g_0 t^2 \tag{3.3}$$

with the width obtained in [1],

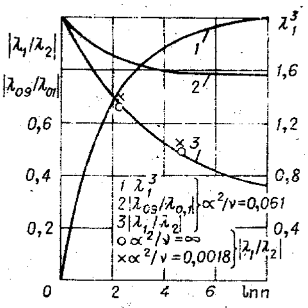

Fig. 3

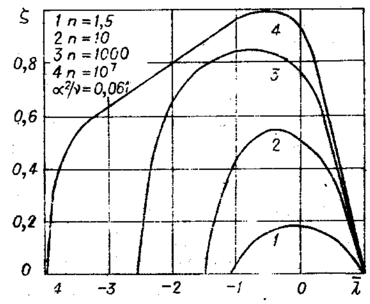

Fig. 4

$$L = 0.084 g_0 t^2 \ln n. \tag{3.4}$$

The last formula is valid for values of n close to unity. In the neighborhood of n = 1 the value of $(\lambda_1 - \lambda_2)^3$ can be described with a good degree of accuracy by the formula

$$(\lambda_1 - \lambda_2)^3 = 9 \ln n. \tag{3.5}$$

Comparing Eqs. (3.3) and (3.4), and taking Eq. (3.5) into account, we then obtain

$$\alpha^2 = 0.084 \ (\alpha = 0.29).$$

We recall that the coefficient in formula (3.4) has been borrowed from [1], where it was determined by the use of experimental data relating to the mixing of jet flow. Thus a somewhat different value of the constant α could be obtained in straight experiments on gravitational mixing. This remark also applies to the derivation of the constant α^2/ν. Its value has been estimated theoretically and needs checking experimentally.

4. Discussion of Results

We note some features of the solution obtained above. Mixing occurs asymmetrically, while the value of $-\lambda_1/\lambda_2$ is basically determined by the initial ratio of densities n and is only weakly dependent on the parameter α^2/ν. The values of $-\lambda_1/\lambda_2$ are given in Fig. 3 by points and crosses, respectively, for $\alpha^2/\nu = \infty$ and $\alpha^2/\nu = 0.018$. For $\alpha^2/\nu = 0.061$ this dependence can be described with a good degree of accuracy by the formula

$$-\lambda_1/\lambda_2 = 1 + 0.222 \ln n.$$

If $n \to 1$, then $-\lambda_1/\lambda_2 \to 1$, i.e., for small density ratios mixing occurs almost symmetrically. If $n \to \infty$, then $-\lambda_1/\lambda_2 \to \infty$, but in such a way that λ_1 remains finite (see Fig. 3). Thus for large density ratios a larger region on the side of the light substance is occupied by mixing, and in the limit its width increases indefinitely. However, if the width is measured according to the region where $0.1 \leq \bar{\delta} \leq 0.9$, then the degree of symmetry, defined by the ratio $-\lambda_{0.9}/\lambda_{0.1}$, is finite for $n \to \infty$ and equal to 0.78 as can be seen from Fig. 3.

It follows from Fig. 2 that for a fixed value of the constant α^2/ν the dimensionless density profile $\bar{\delta}$, measured in units of the dimensionless distance $\bar{\lambda}$, depends on the density ration. This dependence occurs basically for values of n < 10 (curves 2 and 3) and in the region of small densities $\bar{\delta} < 0.1$. If n > 10, then the profiles $\bar{\delta}(\bar{\lambda})$ differ only slightly from each other (curves 1 and 2).

Profiles of $\zeta(\bar{\lambda})$ are given in Fig. 4 for different values of n. It is an interesting fact that as n increases, the maxima of ζ move from the point $\bar{\lambda} = 0$ to the point $\bar{\lambda} = -2.57$ (for n = 1000) and then return to the point $\bar{\lambda} = 0$. The shape of the ζ curve changes substantially in the process.

Thus the mixing process occurs asymmetrically, and the greater the initial density differential n, the larger the asymmetry. The results of [2], where the mixing region is symmetric and the profile $\bar{\delta}(\bar{\lambda})$ independent of n, can be used only in the neighborhood of n = 1 on condition that the empirical constant is chosen on the ground that the profiles agree. Since it was shown in [2] that the solution of [1] differs only slightly from that of [2], then this conclusion also refers to the solution constructed in [1]. However, if we assume (as was done in Sec. 3 in calculating the constant α) that the regions occupied by mixing coincide in [1] and in the present paper, then radically different profiles $\bar{\delta}(\bar{\lambda})$ are obtained for any initial density differential (curve 4 in Fig. 2 is taken from [1] for n = 100). By contrast with [1] calculation of the time derivative in the kinetic energy equation of balance leads to a solution in which the density profiles do not join up smoothly at the mixing fronts.

APPENDIX 1

We shall show that the required solution must emerge from the point I (2.9) and pass to the point II (2.10). To do this we must establish that $y_1 = (2/3)\lambda_1$. We shall consider all the allowable values of y_1: $y_1 = 0$, $y_1 = \infty$, and $y_1 > 0$ finite.

1) $y_1 = 0$. In the neighborhood of the point $(\lambda_1, 0, 0)$ the system of equations (2.8) assumes the form

$$\zeta' = (3/2\lambda_1)[(1/3)\zeta - y^2]; \; y' = -\lambda_1 y/3\zeta.$$

It can be shown that the required solution satisfying the obvious condition $\zeta > 0$ is not among the solutions emerging from the singular point $(\lambda_1, 0, 0)$.

2) $y_1 = \infty$. In this case the system of equations (2.8) can be replaced by the abridged system

$$\zeta' = -(3/2\lambda_1)y^2, \; y' = (3/4\lambda_1)y^3/\zeta.$$

We divide one equation by the other and integrate:

$$dy/d\zeta = -y/2\zeta, \; y = c/\sqrt{\zeta},$$

where c is a constant of integration. We obtain the result that the turbulent flow is nonzero at the mixing front, since

$$lv\partial\rho/\partial x \sim \zeta\delta \approx \zeta y^2 = c,$$

which is in contradiction with physical intuition.

3) $y_1 > 0$ and is constant. The abridged system of equations assumes the form

$$-(2/3\lambda_1 + \zeta') = \zeta(y_1^2 + 2y'/y_1), \quad -2\lambda_1\zeta'/3 = y_1^2.$$

It turns out that a curve emerging from point I and with the necessary characteristics exists only for $y_1 = 2\lambda_1/3$.

Point II is investigated in the same way, and it can be shown that

$$y_2 = -2\lambda_2/3.$$

APPENDIX 2

Let $\nu = 0$. Then, after multiplication by $3\delta\zeta$ and straightforward transformations, the first equation of the system (2.4) assumes the form

$$\delta\zeta^2 - \lambda(\delta\zeta^2)' - 3\zeta\delta' = 0.$$

Substituting $\zeta\delta'$ in the second equation of system (2.4), we obtain

$$2\delta' = (\delta\zeta^2)''.$$

After integration

$$2(\delta - 1) = (\delta\zeta^2)' \text{ for } \lambda > 0, \; 2(\delta - 1/n) = (\delta\zeta^2)' \text{ for } \lambda < 0.$$

Here the boundary conditions (2.5), (2.6) are used to determine the constant of integration. We introduce the symbol

$$z = \zeta^2\delta$$

and obtain, finally,

$$6(\sqrt{\delta})' = \sqrt{z} - 2\lambda(\sqrt{z})',$$
$$z' = \begin{cases} 2(\delta - 1), & \lambda > 0 \\ 2(\delta - 1/n), & \lambda < 0 \end{cases}$$

with the boundary conditions (2.5), (2.6). From this it can be deduced that the function z has a discontinuity at $\lambda = 0$. At the point $\lambda = 0$ the density derivative δ' is continuous. The solution can be constructed numerically. The density profile for n = 10 is given in Fig. 1.

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GAS OSCILLATIONS IN A PIPE WITH A NONLINEAR ACTIVE LOAD

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Characteristics of the working cycle in the cylinder of a piston compressor lead to the appearance of pressure and velocity oscillations of the medium being transported in the associated manifold system, i.e., a pulsating gas flow. Oscillatory gasdynamic processes in pipes lead to an appreciable decrease in efficiency and reliability in the use of a compressor. One of the most efficient methods of decreasing the effect of a pulsating gas flow is the matching of the initial section of the manifold. The essence of the method consists in fitting a lumped resistance (load) after that part of the manifold which is closest to the cylinder. The magnitude of this load is arranged to be equal to the wave resistance of the pipe [1]. Since the gas oscillations are low-frequency, the load resistance turns out to be active and nonlinear as in a steady-state flow [2]. Hence the possibility of matching with a nonlinear resistance needs to be investigated further.

The one-dimensional nonsteady-state motion of a gas in a round pipe of constant cross section with a velocity of motion much less than the velocity of sound is described by the system of equations [3]

$$\dot{\rho} + w' = 0, \ \dot{w} + (w^2/\rho + \rho^\gamma/\gamma)' + (\lambda/\rho)w|w| = 0 \ (0 \leqslant x \leqslant 1). \tag{1}$$

The system (1) has been written in dimensionless form (the length of the pipe, the velocity of sound, and the equilibrium gas density are equal to unity), w is the flow, ρ is the relative density, γ is the polytropic index, and λ is the dimensionless hydraulic resistance coefficient. A point denotes differentiation with respect to time t and a dash, with respect to the x coordinate.

The boundary conditions have the form

$$w = f(\omega t) \ (x = 0); \ (\rho^\gamma - 1)/\gamma = (\eta/\rho)w|w| \ (x = 1), \tag{2}$$

where f is a given periodic function with period 2π, and η is the hydraulic resistance coefficient of the lumped insert in the Darcy–Weissbach formula.

In what follows we shall assume that $\eta \gg 1$, $\lambda \lessdot 1$, while the basic nonlinear effect is associated with the active resistance for x = 1.

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